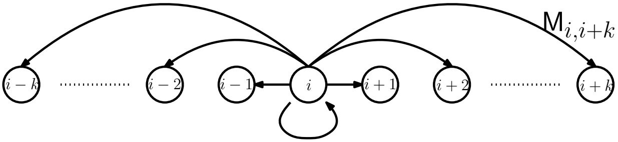
Almost upper directed Markov chains on trees

Luis Fredes
(Work with J.F. Marckert)

Probability and statistical mechanics seminar

Transition matrix: $M = [M_{i,j}]_{i,j \in S}$ with non-negative real entries that sum up to one on each row. A Markov chain Y with transition matrix M satisfies

$$\mathbb{P}(X_{n+1} = b | X_n = a) = M_{a,b}.$$



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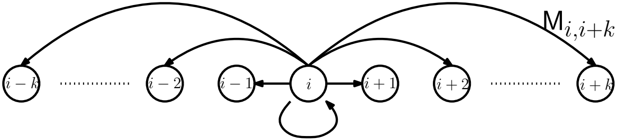


Irreducible: every pair $a, b \in S$ has a finite sequence $\ell = \ell(a, b)$ of steps with positive probability ($M^{\ell}_{a,b} > 0$) such that b can be reached from a.

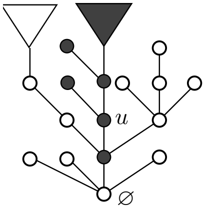
From now on we assume irreducibility

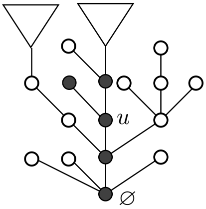
Recurrence / Transience: A chain with transition matrix M is called recurrent if for all/one state $a \in S$ the probability to return to a is one, otherwise the chain is called transient.
Positive recurrence: The expected return time of all/one state $a \in S$ is finite, i.e. $\mathbb{E}_a(\tau_a^+) < +\infty$.
Invariant measure: A measure π on S is said to be invariant by M if it is positive and

$$\sum_{a \in S} \pi_a M_{a,b} = \pi_b \quad \text{for all } b \in S.$$

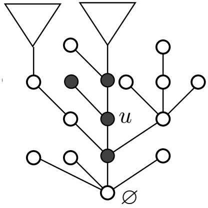
Almost upper and lower directed markov chains on trees

Almost-upper directed ◹ and almost-lower directed ◺

U = 

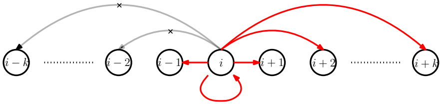
L = 

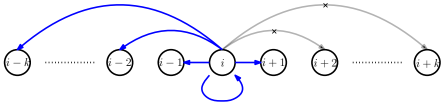
Nearest neighbors random walks are both

NNRW = 

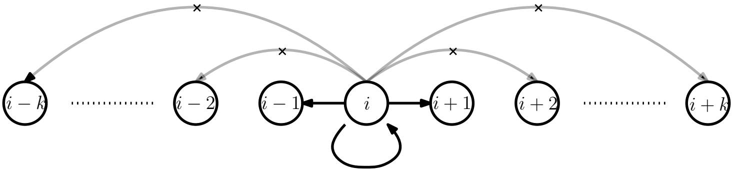
Motivation

Almost-upper directed ◹ and almost-lower directed ◺ on $\mathbb{N}$





Birth and death processes (BDP) are both



Almost-upper directed ◹ and almost-lower directed ◺ on $\mathbb{N}$

$$\color{red}{U := \begin{bmatrix} U_{0,0} & U_{0,1} & U_{0,2} & U_{0,3} & U_{0,4} & \cdots \\ U_{1,0} & U_{1,1} & U_{1,2} & U_{1,3} & U_{1,4} & \cdots \\ 0 & U_{2,1} & U_{2,2} & U_{2,3} & U_{2,4} & \cdots \\ 0 & 0 & U_{3,2} & U_{3,3} & U_{3,4} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \cdots \end{bmatrix}}, \quad \color{blue}{L := \begin{bmatrix} L_{0,0} & L_{0,1} & 0 & 0 & 0 & \cdots \\ L_{1,0} & L_{1,1} & L_{1,2} & 0 & 0 & \cdots \\ L_{2,0} & L_{2,1} & L_{2,2} & L_{2,3} & 0 & \cdots \\ L_{3,0} & L_{3,1} & L_{3,2} & L_{3,3} & L_{3,4} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \cdots \end{bmatrix}}$$

Birth and death processes (BDP)

$$T = \begin{bmatrix} T_{0,0} & T_{0,1} & 0 & 0 & 0 & \cdots \\ T_{1,0} & T_{1,1} & T_{1,2} & 0 & 0 & \cdots \\ 0 & T_{2,1} & T_{2,2} & T_{2,3} & 0 & \cdots \\ 0 & 0 & T_{3,2} & T_{3,3} & T_{3,4} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \cdots \end{bmatrix}$$

Theorem: Birth and death process (Karlin & McGregor '57)

The following measure π with $\pi_0 = 1$

$$\pi_a = \prod_{j=1}^{a} \frac{\mathsf{T}_{j-1,j}}{\mathsf{T}_{j,j-1}} \quad \text{for all } a \geq 1,$$

is the **unique** invariant by T up to a constant factor and the chain is

- **positive recurrent** if and only if

$$\sum_{k\geq 1} \prod_{j=1}^{k} \frac{\mathsf{T}_{j-1,j}}{\mathsf{T}_{j,j-1}} < +\infty$$

- **recurrent** if and only if

$$\sum_{k\geq 0} \prod_{j=1}^{k} \frac{\mathsf{T}_{j,j-1}}{\mathsf{T}_{j,j+1}} = +\infty.$$

Theorem: Almost-upper triangular case on $\mathbb{N}$ (F. -Marckert '21)

The following measure π with $\pi_0 = 1$

$$\pi_a := \frac{\det(\mathrm{Id} - \mathsf{U}_{[0,a-1]})}{\prod_{j=1}^{a} \mathsf{U}_{j,j-1}} \quad \text{for all } a \geq 1.$$

is the **unique** invariant by U up to a constant factor and the chain is

• **positive recurrent** if and only if π_u could be renormalize into a probability distribution.

• **recurrent** if and only if

$$\lim_{b \to +\infty} \mathsf{U}_{1,0} \frac{\det(\mathrm{Id} - \mathsf{U}_{[2,b-1]})}{\det(\mathrm{Id} - \mathsf{U}_{[1,b-1]})} = 1.$$

Upper directed auxiliary Kernel on a tree

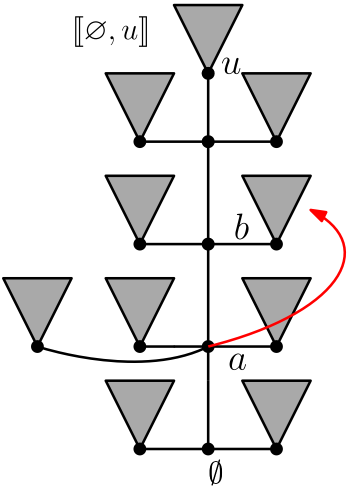


Figure: Construction of the auxiliary transition matrix.

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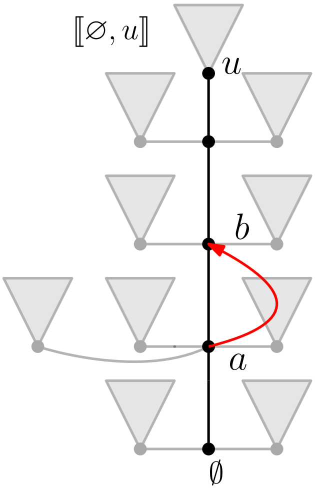

Figure: Transitions from nodes on $[\![\varnothing, u]\!]$ are redirected to the common ancestor with u.

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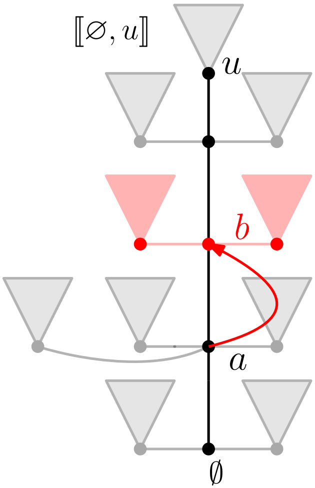

Figure: The total mass associated to the transition.

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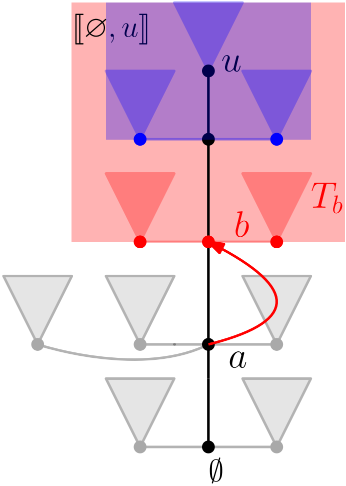

Figure: The total mass as a difference of two trees.

Upper directed auxiliary Kernel on a tree

Define :

- The parent of u as $p(u)$.
- The set $[\![\varnothing, u]\!]$ of nodes belonging to the unique path from $\varnothing$ to u.
- For a subset of nodes B, the mass going to B as $\mathrm{U}_{a,B} = \sum_{b\in B} \mathrm{U}_{a,b}$.
- The tree T_a as the subtree of T rooted at a.
- For a node u in T and $v \in [\![\varnothing, p(u)]\!]$, the successor of v in the direction of u is denoted by $\mathrm{s}(v, u)$

Consider the weighted graph $({}^uG, {}^u\mathrm{U})$ with set of nodes ${}^uV := [\![\varnothing, u]\!]$, and in which the weight of

$${}^u\mathrm{U}_{a,b} = (\mathrm{U}_{a,T_b} - \mathrm{U}_{a,T_{\mathbf{s}(b,u)}})\mathbf{1}_{b\in[\![p(a),p(u)]\!]} + \mathbf{1}_{b=u}\mathrm{U}_{a,T_u}$$

is defined for all $a \in [\![\varnothing, p(u)]\!]$ and $b \in [\![\varnothing, u]\!]$

Theorem: Almost upper directed MC on trees (F. -Marckert '24)

The following measure π that we call h-invariant with $\pi_\varnothing = 1$

$$\pi_u := \frac{\det(\mathrm{Id} - {}^u\mathrm{U}_{[\![\varnothing, p(u)]\!]})}{\prod_{v \in]\!]\varnothing, u]\!]} \mathrm{U}_{v,p(v)}} \quad \text{for all } u \in T.$$

is **one** invariant measure of U and the chain is

- **positive recurrent** if and only if π could be renormalize into a probability distribution.
- **recurrent** if and only if for any children i of $\varnothing$

$$\lim_{b \to +\infty} \mathrm{U}_{i,\varnothing} \frac{\det(\mathrm{Id} - \mathrm{U}^{(i)}_{[i,<b]})}{\det(\mathrm{Id} - \mathrm{U}_{[i,<b]})} = 1.$$

where $U_{[i,<h]}$ is the restriction of U to the subtree T_i to the $h-1$ first levels of T.

Remark : In the case of recurrence the h-invariant is the unique invariant measure up to constant.

Plan of the talk

- Proof 1/2 : the h-invariant is an invariant measure for U.
- How to compute the h-invariant at a given point.
- The h-invariant as an invariant measure of a conditioned Markov chain.
- Discussion about ends.
- Spectral properties.

Combinatorial warm up I: Matrix tree theorem

ST(G)= set of spanning trees of G.

Matrix-tree theorem [Kirchhoff]

$$|\text{ST}(G)| = \det\left(\text{Laplacian}_G^{(r)}\right),$$

where $\text{Laplacian}_G^{(r)}$ is the Laplacian matrix of G deprived of the line and column associated to r.

$$\text{Laplacian}_G(i,j) = \left[deg(u_i)\mathbb{1}_{i=j} - |\{u_i, u_j\} \in E|\right]$$

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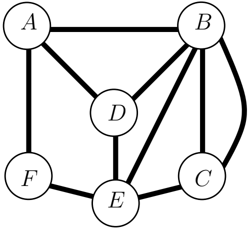


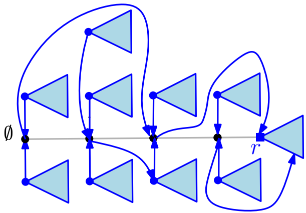
$$\mathrm{Laplacian_G} = \begin{pmatrix} 3 & -1 & 0 & -1 & 0 & -1 \\ -1 & 5 & -2 & -1 & -1 & 0 \\ 0 & -2 & 3 & 0 & -1 & 0 \\ -1 & -1 & 0 & 3 & -1 & 0 \\ 0 & -1 & -1 & -1 & 4 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$|\mathrm{ST}(G)| = \det\left(\mathrm{Laplacian}_G^{(A)}\right) = 98.$$

Canonical orientation on rooted trees : all the edges pointing towards the root. Consequence : every vertex but the root has exactly one outgoing edge.

How does a spanning tree of the allowed transitions look like?

Here we work on the graph G whose set of vertices are the same vertices of the tree T and whose set of edges are all having positive transition probability with respect to U.



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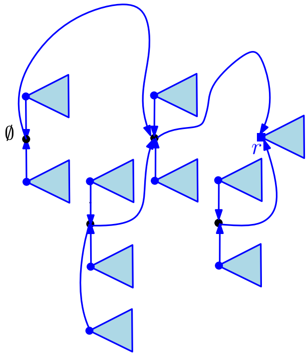


ST(G,r):= set of spanning trees τ of G **(finite graph)** rooted at r.
$W_M(\tau,r) := \prod_{\vec{e}\in\tau} M_{\vec{e}}$ with edges pointing towards the root r.

Weighted Matrix-tree theorem

$$\sum_{\tau\in \mathrm{ST}(G,r)} W_M(\tau,r) = \det\left(\mathrm{Id} - M^{(r)}\right),$$

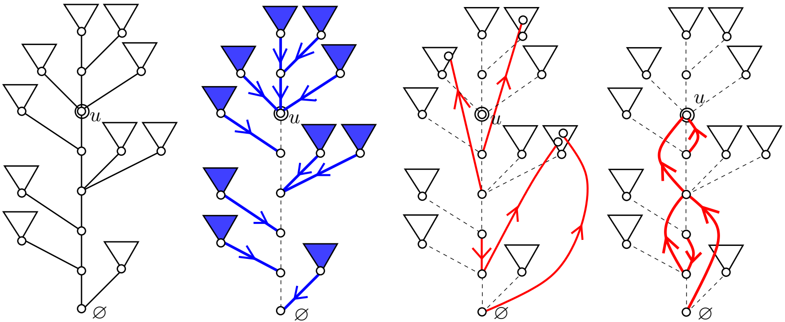
where $M^{(r)}$ is the matrix M deprived of the line and column r.

Markov chain tree theorem

The invariant probability measure ρ of M satisfies

$$\rho_v \overset{\text{(Cramer)}}{=} \frac{\det(I - M^{(v)})}{Z} \overset{\text{(WMTT)}}{=} \frac{\sum_{\tau\in \mathrm{ST}(G,v)} W_M(\tau,v)}{Z}$$

We characterize weighted rooted spanning trees τ on the graph of the Markov chain.



A spanning tree τ seen from each vertex gives:

- If a vertex $v \notin [\![\varnothing, u]\!]$, then it is forced to take the edge $(v, p(v))$. These edges generates a forest.
- Each vertex $v \in [\![\varnothing, u]\!]$ joins a connected component of the forest and it reaches another vertex in $[\![\varnothing, u]\!]$.
- We contract the connected components of the forest. This translates the weighted trees of red edges as the determinant of the Laplacian matrix of ${}^{u}\mathrm{U}$.

Proof 1/2 : h-invariant for finite trees

So the weight of spanning trees rooted at u is given by

$$\sum_{\tau \in \mathsf{ST}(G,r)} W_{\mathsf{U}}(\tau, r) = \det(\mathrm{Id} - {}^{u}\mathsf{U}_{[\![\varnothing, p(u)]\!]}) \prod_{v \notin [\![\varnothing, u]\!]} \mathsf{U}_{v,p(v)} = C \frac{\det(\mathrm{Id} - {}^{u}\mathsf{U}_{[\![\varnothing, p(u)]\!]})}{\prod_{v \in]\!]\varnothing, u]\!]} \mathsf{U}_{v,p(v)}}$$

From the Markov Chain tree theorem one has that the (unique up to a constant) invariant measure π_u of the chain U at u is proportional to the weight of spanning trees τ rooted at u.

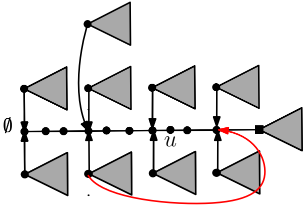
Now for infinite trees, the balance equations give

$$\rho_u = \sum_{v \in [\![\varnothing, v]\!]} \rho_v \mathrm{U}_{v,u} + \sum_{v \in \mathrm{children}_T(u)} \rho_v \mathrm{U}_{v,u} \qquad (1)$$

We show that π_u satisfies this system with an auxiliary irreducible Markov chain on a finite tree. The finite tree is the restriction of the tree T until generation $|u|+2$ and its transition matrix $\mathrm{U}|$ coincides with U for transitions up to $|u|+1$

$$\mathrm{U}|_{u,v} = \begin{cases} \mathrm{U}_{u,v} & \text{if } |v| < |u|+2 \\ \mathrm{U}_{u,T_v} & \text{if } |v| = |u|+2 \end{cases}$$

And from the previous part π_u solves the balance equation (1).



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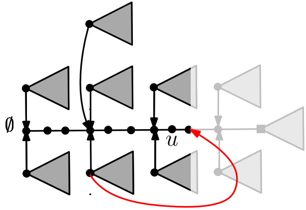


The leaf addition strategy

Let τ and τ' be two infinite or finite trees such that τ' can be obtained from τ by the suppression of a leaf ℓ. Assume that U and U$'$ are two ◹ on τ and τ' satisfying that :

$$\mathrm{U}'_{i,j} = \begin{cases} \mathrm{U}_{i,j} & \text{if } i \in \tau', j \in \tau' \setminus \{p(\ell)\} \\ \mathrm{U}_{i,p(\ell)} + \mathrm{U}_{i,\ell} & \text{if } i \in \tau', j = p(\ell) \end{cases}$$

Proposition

If ρ' is a 1-left eigenvector of U$'$*, then ρ given by*

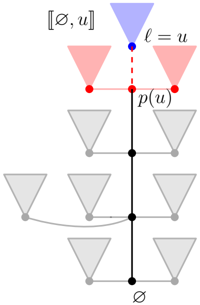
$$\rho_i = \begin{cases} \rho'_i & \text{if } i \neq \ell \\ \sum_{v \in [\![\varnothing, p(\ell)]\!]} \rho'_v \frac{\mathrm{U}_{v,\ell}}{\mathrm{U}_{\ell,p(\ell)}} & \text{if } i = \ell \end{cases}$$

is a 1-left eigenvector of U*.*

From our proof we see that ${}^{u}\mathrm{U}$ has the same invariant information that U on $[\![\varnothing, u]\!]$

From this we see that we can grow our tree from $\tau' = [\![\varnothing, p(u)]\!]$ using $\mathrm{U}' = {}^{p(u)}\mathrm{U}$ to $\tau = [\![\varnothing, u]\!]$ with $\mathrm{U} = {}^{u}\mathrm{U}$ and they satisfy the leaf addition requirements. This translates into a recursive way to find the value of ρ_u given $\rho_{[\![\varnothing, p(u)]\!]}$.

$$\rho_u = \sum_{v \in [\![\varnothing, p(u)]\!]} \rho_v \frac{\mathrm{U}_{v, T_u}}{\mathrm{U}_{v, p(v)}}$$



Example

Consider a finite tree T and a Markov chain such that starting from u we jump to the parent (when available) or a descendant uniformly at random. We set $\rho_\varnothing = 1$. From the leaf addition applied to $[\![\varnothing, u]\!]$ for $u \neq \varnothing$ we obtain

$$\rho_u = \sum_{v \in [\![\varnothing, p(u)]\!]} \rho_v \frac{\mathrm{U}_{v,T_u}}{\mathrm{U}_{u,p(u)}} = \sum_{v \in [\![\varnothing, p(u)]\!]} \rho_v \frac{|T_u|/(\mathbb{1}_{v \neq \varnothing} + |T_v|)}{1/(1+|T_u|)}$$

Now we apply recursively this formula on ρ_v to obtain

$$\rho_u = \frac{|T_u|}{|T_\varnothing|} \left[\prod_{v \in]\!]\varnothing, u]\!]} (1 + |T_v|) \right]$$

Let $t \subset T$ a finite subtree of the tree T. Define

$$h_t(u) = \mathbb{P}_u(\tau_t < \infty)$$

Define the Doob's h- transform of the matrix U as the matrix $\tilde{\mathrm{U}}(t)$

$$\tilde{\mathrm{U}}(t)_{i,j} = \begin{cases} \mathrm{U}_{i,j} & \text{if } i \in t \\ \frac{h_t(j)}{h_t(i)} \mathrm{U}_{i,j} & \text{if } i \notin t \end{cases}$$

This is the transition matrix of a path conditioned to come back to t.

- The invariant measure π^t of $\tilde{\mathrm{U}}(t)$ must coincide with π on the interior of t.
- The invariant measure π^t is unique since $\tilde{\mathrm{U}}(t)$ is recurrent.

Conclusion : The measure $\pi^{t_n}(u) \to \pi(u)$ as $t_n \to T$ for every $u \in T$.
Meaning that the chain come back to any finite tree

Take home message : the h-measure π of U is the invariant measure of U when "forced to be recurrent".

Discussion about recurrence and positive recurrence

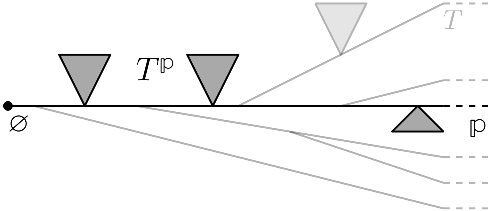
When the tree is $\mathbb{N}$, the system is given by a almost-upper triangular transition matrix, whose balance equations are given by

$$\pi_b = \sum_{a \leq b+1} \pi_a \mathsf{U}_{a,b} \iff \pi_{b+1} = (\pi_b - \sum_{a \leq b} \pi_a \mathsf{U}_{a,b})/\mathsf{U}_{b+1,b}$$

So in this case the invariant measure is unique (up to a constant factor)

What can we say about ends?

For an infinite path $\mathbb{p}$ we define the transition matrix $\mathsf{U}^{\mathbb{p}}$ acting on the tree $T^{\mathbb{p}}$ consisting on $\mathbb{p}$ together with the finite subtrees hanging from $\mathbb{p}$, such that the jumps of U outside $T^{\mathbb{p}}$ (one ended tree) are considered in $\mathsf{U}^{\mathbb{p}}$ as going towards the first common ancestor on $T^{\mathbb{p}}$.



Proposition

An irreducible $\triangledown$ *transition matrix* U *on an infinite tree* T *with a finite number of ends is:*

- *recurrent* $\iff$ *for every end* $\mathbb{p}$, *the matrix* $\mathrm{U}^{\mathbb{p}}$ *is recurrent.*
- *positive recurrent* $\iff$ *for every end* $\mathbb{p}$, *the matrix* $\mathrm{U}^{\mathbb{p}}$ *is positive recurrent.*

Remark :

- The positive recurrence cares about finite trees, since the chain may spend a lot of time on them.
- To study recurrence, one can prune the finite subtrees and direct all transitions to the first ancestor in an infinite path. The resulting graph is a tree composed with only infinite paths.

What about infinite number of ends?

Lemma

Let U *a* $\triangledown$ *transition matrix on* T *an infinite tree. If there is an end* $\mathbb{p}$ *with* $\mathsf{U}^{\mathbb{p}}$ *transient, then* U *is transient.*

The reciprocal is not true : we give an example in the complete infinite binary tree, where $\mathsf{U}^{\mathbb{p}}$ is positive recurrent for every end $\mathbb{p}$, but the chain U is transient.

Spectral properties

Is the number of invariant measures related with the number of ends having a transient $\mathbb{U}^{\mathbb{P}}$?

...Long story short : The 1-left eigenspace has dimension related with the number of bifurcations in the tree induced by just ends (when dropping finite trees). This bounds the number of extremal invariant measures (only the positive ones are kept).

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Can we say something about other eigenvalues?

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Can we say something about other eigenvalues?

Proposition

Let U *be a* ◹ *irreducible transition matrix on a tree* T *with no leaves. For all* $\lambda \in \mathbb{C}$*, the vector* $\pi^{(\lambda)}$ *defined by* $\pi^{(\lambda)}(\varnothing) = 1$ *and*

$$\pi^{(\lambda)}(u) := \pi^{(\lambda)}(\varnothing)\frac{\det\left((\lambda \mathrm{Id} - {}^{u}\mathsf{U})^{(u)}\right)}{\prod_{v \in \rrbracket \varnothing, u \rrbracket} \mathsf{U}_{v,p(v)}}, \quad \text{for } u \in T$$

is a λ*-left eigenvector of* U*; hence the spectrum of* U *is* $\mathbb{C}$*.*

Thank you for your attention!

Invariant measure and positive recurrence criteria: tridiagonal case
The uniqueness of the invariant measure with $\pi_0 = 1$ gives the equality of two formulas for the invariant mesure (Karlin-McGregor and F.-Marckert)

$$\frac{\det(\mathsf{Id} - \mathsf{T}_{[0,a-1]})}{\prod_{j=1}^{a} \mathsf{T}_{j,j-1}} = \prod_{j=1}^{a} \frac{\mathsf{T}_{j-1,j}}{\mathsf{T}_{j,j-1}} \qquad \forall a \geq 1$$

then, the positive recurrence criteria is recovered.

Recurrence criteria

Set $D_{i,j} = \det(\mathrm{Id} - \mathsf{T}_{[i,j]})$, then

$$D_{i,j} = (1 - \mathsf{T}_{i,i})D_{i+1,j} - \mathsf{T}_{i,i+1}\mathsf{T}_{i+1,i}D_{i+2,j}, \tag{2}$$

and set

$$Z_{i,j} := \frac{D_{i,j}}{D_{i+1,j}\mathsf{T}_{i,i-1}}$$

and notice that our result translates into: the MC with transition matrix T is recurrent iff

$$\lim_{b\to+\infty} \mathsf{T}_{1,0}\frac{\det(\mathrm{Id} - \mathsf{T}_{[2,b-1]})}{\det(\mathrm{Id} - \mathsf{T}_{[1,b-1]})} = \lim_{b\to+\infty} \frac{1}{Z_{1,b-1}} = 1.$$

(2) rewrites

$$Z_{i,j} = \frac{(1 - \mathsf{T}_{i,i})}{T_{i,i-1}} - \frac{\frac{\mathsf{T}_{i,i+1}}{\mathsf{T}_{i,i-1}}}{Z_{i+1,j}},$$

which gives the convergents of a continued fraction.

$$Z_{1,b-1} := c_1 + \cfrac{a_2}{c_2 + \cfrac{a_3}{\ddots + \cfrac{\ddots}{c_{b-2} + \cfrac{a_{b-1}}{c_{b-1}}}}}$$

These can be solved and give that

$$Z_{1,b-1} = 1 + \left(\sum_{k=1}^{b-1} \prod_{j=1}^{k} \frac{\mathsf{T}_{j,j-1}}{\mathsf{T}_{j,j+1}} \right)^{-1}$$

So that $\lim_{b\to\infty} Z_{1,b-1} = 1$ is equivalent to Karlin & McGregor's recurrence criteria.

$$\sum_{k\geq 0} \prod_{j=1}^{k} \frac{\mathsf{T}_{j,j-1}}{\mathsf{T}_{j,j+1}} = +\infty.$$